

On Multiparticle Entanglement via Resonant Interaction between Light and Atomic Ensembles

V.N. Gorbachev, A.I. Trubilko

*Laboratory for quantum information & computation, University of AeroSpace Instrumentation,
St-Petersburg, 190000, Bolshaia Morskaia 67, Russia.*

Multiparticle entangled states generated via interaction between narrow-band light and an ensemble of identical two-level atoms are considered. Depending on the initial photon statistics, correlation between atoms and photons can give rise to entangled states of these systems. It is found that the state of any pair of atoms interacting with weak single-mode squeezed light is inseparable and robust against decay. Optical schemes for preparing entangled states of atomic ensembles by projective measurement are described.

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I. INTRODUCTION

Now properties of multiparticle entangled states, their preparation and application are the subject of extensive discussion. The desired state of a physical system can be prepared either by projective measurement or as a result of evolution. For atomic systems, both methods have already been implemented in experiments. In particular, two atomic ensembles were used in [1] to create an EPR pair by projective measurement. The latter method was demonstrated in several studies: entangled states of alkali ions were generated via Coulomb interaction [2], neutral Rydberg atom were used to create an EPR pair in a micromaser setup [3], and resonant dipole - dipole interaction was used for entangling neutral atoms in an optical lattice [4]. The most popular methods for preparing entangled photon states are still mostly based on parametric down-conversion. For example, an entangled state equivalent to a three-state quantum system (qutrit) was prepared and examined by using quantum state tomography in [5]. These examples suggest that an entangled state of two systems can be prepared experimentally by using a certain interaction. Systems of this kind are well studied. With regard to applications, it is important to know how entanglement can be utilized and its robustness against decoherence. In this respect, of special interest are multiparticle systems, whose entangled states are characterized by much more complicated and diverse behavior.

Previous efforts were mainly focused on analysis of entanglement between several particles. In particular, the W class of tripartite entanglement defined in [6] includes the symmetric three-photon polarization entangled state implemented in the experiment reported in [7]. An extension to four qubits was proposed in [8], where nine inequivalent classes were distinguished that cannot be connected by local operations and quantum communication. Studies of multiparticle systems are relatively few, being focused on entanglement criteria and application to problems in quantum information theory. Whereas the Peres - Horodecki criterion for bipartite entanglement found in [9] was applied to a real physical system in [10], no operational criterion is known for entanglement in the general case, and various approaches are often used. In [11], the concept of entanglement molecules [12] was introduced to propose a classification using graphs, in which particles and classical or quantum correlations represented, respectively, by vertices and edges connecting pairs of vertices. Graphs of this kind can be used to describe both pure and mixed entangled states and distinguish several classes differing by topological properties of the graphs. In [13], symmetric states (including Dicke states) were studied by using several entanglement measures (entanglement entropy, negativity, and entanglement of formation) defined by the eigenvalues of a partial transpose of the density matrix. A numerical analysis was performed to find that symmetric states are robust to particle loss even if the number of particles is large (up to 10^3). Note that the calculation of eigenvalues is a difficult task, because the dimension of an ensembles Hilbert space exponentially increases with the number of constituent particles. Owing to their robustness, symmetric states can be used in such applications as cloning and telecloning protocols for quantum information transmission [14], quantum key distribution [15], and quantum teleportation or dense coding [16]. The formulation of two models of a one-way quantum computer using measurements on multiparticle entangled states [17], [18] has strongly stimulated studies of the properties of multiparticle systems, in particular, Ising- and Bose - Hubbard-like models.

The present study focuses on the Dicke states arising as a result of collective interaction of many atoms with electromagnetic field [19], which has been analyzed in numerous studies (e.g., see [20]). This system exhibits many physical properties of interest for quantum information processing. Photon trapping in chain configurations of atoms was considered in [21]. When the system is placed in a cavity, this effect reduces the photon escape rate and increases

the decoherence time of the cavity mode. In [22], this effect was used for generating W states and anticloning [23], which can be implemented with high fidelity by means of photon trapping. In those studies, only single-photon traps and single-photon initial states were analyzed. Here, we consider the more general case of multiphoton processes, assuming that the photon statistics is arbitrary.

The main questions addressed below are the following: what types of entangled states are produced by interaction between atoms and field? What states can be prepared from independent atomic ensembles entangled with a photon? How can these states be utilized? We consider resonant interaction between narrow-band light and an ensemble of identical two-level atoms coupled to a common heat bath. The analysis is restricted to a simple model of radiative decay. Multiphoton processes, such as Raman scattering, are described in terms of effective Hamiltonians, which can be obtained by unitary transformation [24]. The behavior of an atomic system interacting with light characterized by arbitrary photon statistics is analyzed by using perturbation theory in the interaction strength for arbitrary statistics of light particularly for Gaussian, coherent, and squeezed states. We find that weak single-mode squeezed light is required to create multiparticle entanglement between atoms. As distinct to the case considered in [25], the steady state discussed here is robust against atomic decay. When decay is neglected and analysis is restricted to a single-photon initial state, simple exact solutions describing exchange of excitation between the field mode and atoms can be obtained [26]. These solutions can be used for generating and transforming symmetric Dicke states and for processing and storing quantum information. The optical schemes for projective measurement considered here can be used to generate entangled states of atomic ensembles. An EPR entangled pair of macroscopic ensembles was created in an experiment [1]. The new states produced in our schemes have hierarchical structure, thus differing from the cluster states introduced in [27] as a resource for one-way computing.

The paper is organized as follows. First, we formulate a basic model and write out the second-order perturbation solutions obtained by taking into account radiative decay. These solutions are then used to analyze the states of the atomic system corresponding to various photon statistics. Exact solutions obtained under certain initial conditions by neglecting radiative decay are used to describe generation and transformation of symmetric Dicke states. Finally, we consider optical schemes for preparing entangled states of atomic ensembles by projective measurement.

II. BASIC EQUATIONS

In the dipole approximation, the ensemble of N identical, but distinguishable, two-level atoms interacting with electromagnetic field is described by the Hamiltonian

$$H = i\hbar\vartheta, \\ \vartheta = \sum_k g_k a_k S_k^\dagger - h.c.,$$

where $g_k = (\hbar\omega_k/2\varepsilon_0 L^3)^{1/2}(\mu, e_k)$ is the coupling constant, μ is the dipole transition matrix element, e_k is the polarization vector for the mode with wave vector k , a_k and a_k^\dagger are photon creation and annihilation operators, $S_k^\dagger = \sum_a s_{10}(a) \exp(ikr_a)$ is the atomic operator for the atom located at a point r_a ($x, y = 0, 1$, where 0 and 1 denote the ground and excited states, respectively). When analysis is restricted to interaction with a single resonant mode, S_k can be replaced with $S_{k=0}$, which makes it possible to treat an atomic ensemble occupying a spatial region as a point like object. Then

$$\vartheta = S_{10}B - S_{01}B^\dagger, \quad (1)$$

where $S_{10} = \sum_a |1\rangle_a \langle 0|$, $B = ga$. Effective Hamiltonian (1) is used here to describe not only interaction with a single resonant mode, but also multiphoton processes, such as Raman scattering. In the latter case, we set $B = fa_A a_S^\dagger$, and assume that the photon frequencies ω_A and ω_S satisfy the relation $\omega = \omega_A - \omega_S$, where ω is the atomic transition frequency. Hamiltonians of this form can be obtained by unitary transformation [24].

The density matrix ρ of N atoms interacting with field obeys the master equation

$$\frac{\partial}{\partial t}\rho = [\vartheta, \rho] + \mathcal{L}\rho, \quad (2)$$

where relaxation is represented by the Lindblad superoperator

$$\mathcal{L} = \sum_a \mathcal{L}_a,$$

$$\mathcal{L}_a = -\frac{\gamma_{\uparrow}}{2}[s_{01}(a)s_{10}(a)\rho - s_{10}(a)\rho s_{01}(a)] - \frac{\gamma_{\downarrow}}{2}[s_{10}(a)s_{01}(a)\rho - s_{01}(a)\rho s_{10}(a)] + h.c. \quad (3)$$

This representation corresponds to the model of purely radiative decay with longitudinal and transverse decay rates $\gamma = \gamma_{\downarrow} + \gamma_{\uparrow}$ and γ_{\perp} , which satisfy the relation $\gamma_{\perp} = \gamma/2$. In general case $\gamma_{\perp} > \gamma/2$ since γ_{\perp} should be replaced by $\gamma_{\perp} + \kappa$, where κ is a dephasing collision rate.

Effective Hamiltonian (1) may involve many field modes with ω_k differing from the atomic transition frequency by $\delta\omega_k$ and occupying a frequency band of width $\Delta\omega$. If $\Delta\omega, \delta\omega_k \ll \gamma_{\perp}$, then we can consider a narrow-band radiation field and make use of resonance approximation. Otherwise, the field must be described in terms of multiple-time correlation functions. Solution of Eq. (2) is a difficult task. To describe the interaction between single atom and field, the following equation for the density matrix $\rho_a = Tr'_{a'}\rho$ is derived from (2) by tracing over all atoms except for one:

$$\frac{\partial}{\partial t}\rho_a = [\vartheta_a, \rho_a] + \mathcal{L}_a\rho_a + N(N-1)Sp_{a'}[\vartheta_{a'}, \rho_{aa'}], \quad (4)$$

where $\vartheta_a = s_{10}(a)B - h.c.$ and $\rho_{aa'} = Sp'_{aa'}\rho$ is a two-particle density matrix. The right-hand side of (4) contains a multiparticle contribution proportional to $N(N-1)$, because the density matrix $\rho_{aa'}$ does not commute with the field operators. This leads to the Bogolyubov- Born-Green-Kirkwood-Yvon chain of equations for the multiparticle density matrices $\rho_a, \rho_{aa'}, \rho_{aa'a''}, \dots$. In physical terms, this means that fluctuations of quantized electromagnetic field induce correlation between atoms. If the field is assumed to be classical and noise-free, for example, a coherent state is considered, then the interaction will not give rise to any correlation, and the initially uncorrelated atoms will remain mutually independent. In what follows, we use (2) to analyze interactions that can be used to generate symmetric Dicke states.

III. DICKE STATES

First, we define symmetric Dicke states and introduce a representation of symmetric Dicke states that demonstrates their relation to the collective interaction processes. The Dicke states are eigenstates of the operators J_z and $J^2 = J_x^2 + J_y^2 + J_z^2$

$$\begin{aligned} J_z|jma\rangle &= m|jma\rangle \\ J^2|jma\rangle &= j(j+1)|jma\rangle, \end{aligned} \quad (5)$$

where $[J_s, J_p] = i\epsilon_{spd}J_d$. For example, operators $J_s, s = x, y, z$ can be represented by Pauli matrices $J_s = (1/2)\sum_k \sigma_{sk}$, $\sigma_{sk}, s = x, y, z$. Indexes j and m are integer or half-integer numbers $|m| \leq j, \max j = N/2$. If $j = N/2$, then the states are symmetric, and the quantum number a introduced to lift degeneracy can be omitted. For h excited atoms $h = m + N/2$, the states can be represented as

$$|j = N/2, m\rangle \equiv |h; N\rangle = \sum_z P_z |1_1, 1_2, \dots, 1_h, 0_{h+1}, \dots, 0\rangle, \quad (6)$$

where P_z is one of the $C_h^N = N!/(h!(N-h)!)$ distinguishable permutations of particles.

The vector $|h; N\rangle$ describes an atomic of h excited atoms and it is normalized as $\langle h; N|h; N\rangle = C_h^N$. Symmetric states of a multiparticle system arise when interaction is described by collective operators of the form $S_{10} = \sum_a^N |1\rangle_a \langle 0|$.

$$|h; N\rangle = (1/h!)S_{10}^h|0; N\rangle. \quad (7)$$

If $h = 1$, then one finds that

$$|1; N\rangle = |10\dots 0\rangle + \dots + |00\dots 1\rangle. \quad (8)$$

Since the wavefunction $|h; N\rangle$ is not factorizable, it represents an entangled state. In terms of correlation between particles, it is substantially different from other entangled states. For example, in the Greenberger HorneZeilinger (GHZ) state $GHZ = (1/2)(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$, the correlation of any M particles ($M < N$) is classical. In particular, the density matrix corresponding to the state $|1; N\rangle\langle 1; N|$ of a group of M particles is $\rho(M \leq N) = N^{-1}|1; M\rangle\langle 1; M| + (N-M)N^{-1}|0; N\rangle\langle 0; N|$. The corresponding von Neumann entropy depends on the relative particle number $p = M/N$: $S(\rho(M \leq N)) = -p \log p - (1-p) \log(1-p)$. When $p = 1/2$ the entropy achieves its maximum 1. If $M = 2$ we can apply the necessary and sufficient separability criterion proposed in [9]. According to this criterion, the state is inseparable (entangled) if the density matrix partially transposed over the one of the atoms has at least one negative

eigenvalue. In the case considered here, one of the four eigenvalues $\{1/N; 1/N; (N-2)(2N)^{-1}[1 \pm \sqrt{1+4/(N-2)^2}]\}$ is negative. Note that the behavior of correlation between M particles depends on $p = M/N$. As the total particle number N increases, $p \rightarrow 0$ and the correlation vanishes, since their state becomes pure as $\rho(M \leq N) \rightarrow |0; N\rangle\langle 0; N|$. In what follows, we make use of the following equalities:

$$\begin{aligned} S_{01}|0; N\rangle &= 0, \\ S_{10}|h; N\rangle &= (h+1)|h+1; N\rangle, \\ S_{01}|h; N\rangle &= (N-h+1)|h-1; N\rangle, \\ S_{01}S_{10}|h; N\rangle &= (h+1)(N-h)|h; N\rangle, \\ S_{10}S_{01}|h; N\rangle &= h(N-h+1)|h; N\rangle. \end{aligned} \tag{9}$$

IV. SECOND-ORDER PERTURBATION THEORY

To solve Eq. (2), we use perturbation theory in the interaction strength:

$$\rho = \rho^{(0)} + \rho^{(1)} + \rho^{(2)} + \dots, \tag{10}$$

Here, the zeroth-order approximation $\rho^{(0)}$ is the steady-state solution of (2) with $\vartheta = 0$: $\rho^{(0)} = |0\rangle\langle 0| \otimes \rho_f$, where the density matrix ρ_f represents the modes and $|0\rangle = |0\rangle^{\otimes N}$ corresponds to the ground state of all atoms. The operators $\rho^{(k)}$, $k = 1, \dots$ satisfy the equations

$$\frac{\partial}{\partial t}\rho^{(k)} = [\vartheta, \rho^{(k-1)}] + \mathcal{L}\rho^{(k)}, \tag{11}$$

subject to the initial conditions $\rho^{(k)}(0) = 0$.

The analysis that follows is restricted to second-order perturbation theory, which is sufficient to obtain statistical characteristics of the excitation field. The matrix equation for $\rho^{(2)}$ is

$$\begin{aligned} \langle 1_k, 1_m; N | \frac{\partial}{\partial t} \rho^{(2)} | 0; N \rangle &= -2\gamma_{\perp} \langle 1_k, 1_m; N | \rho^{(2)} | 0 \rangle + \langle 1_k, 1_m; N | R | 0 \rangle \\ \langle 1_k; N | \frac{\partial}{\partial t} \rho^{(2)} | 1_m; N \rangle &= -2\gamma_{\perp} \langle 1_k; N | \rho^{(2)} | 1_m; N \rangle + \langle 1_k; N | R | 1_m; N \rangle, \quad k \neq m \\ \langle 1_k; N | \frac{\partial}{\partial t} \rho^{(2)} | 1_k; N \rangle &= -\gamma \langle 1_k; N | \rho^{(2)} | 1_k; N \rangle + \langle 1_k; N | R | 1_k; N \rangle \\ \langle 0; N | \frac{\partial}{\partial t} \rho^{(2)} | 0; N \rangle &= \gamma \sum_k \langle 1_k; N | \rho^{(2)} | 1_k; N \rangle + \langle 0; N | R | 0; N \rangle. \end{aligned} \tag{12}$$

where $s_{10}(k)|0; N\rangle = |1_k; N\rangle$ and $s_{10}(k)s_{10}(p)|0; N\rangle = |1_k, 1_p; N\rangle$ represent the states in which only the k th atom is excited and only the k th and p th atoms are excited, respectively. The nonzero matrix elements of the operator $R = [\vartheta, \rho^{(1)}]$ are

$$\begin{aligned} \langle 1_k, 1_m; N | R | 0; N \rangle &= 2\kappa(t)B^2\rho_f, \\ \langle 0; N | R | 0; N \rangle &= -\kappa(t)N(B^{\dagger}B\rho_f + \rho_f B^{\dagger}B), \\ \langle 1_k; N | R | 1_m; N \rangle &= 2\kappa(t)B\rho_{sf}B^{\dagger}, \end{aligned} \tag{13}$$

where $\kappa(t) = (1/\gamma_{\perp})(1 - \exp(-\gamma_{\perp}t))$. For purely radiative decay, $\gamma_{\perp} = \gamma/2$ and the second-order perturbation theory yields

$$\begin{aligned} \rho &= |0\rangle\langle 0| \otimes \rho_f + \kappa[|1; N\rangle\langle 0; N| \otimes B\rho_{sf} + h.c.] + \kappa^2[|2; N\rangle\langle 0; N| \otimes B^2\rho_f + h.c.] \\ &\quad - N\gamma\mathcal{K}|0; N\rangle\langle 0; N| \otimes [B^{\dagger}B\rho_f - B\rho_f B^{\dagger} + h.c.] - (1/2)N\kappa^2|0; N\rangle\langle 0; N| \otimes [B^{\dagger}B\rho_f + h.c.] \\ &\quad + \kappa^2|1; N\rangle\langle 1; N| \otimes B\rho_f B^{\dagger}, \end{aligned} \tag{14}$$

where

$$\mathcal{K} = \gamma_{\perp}^{-1} \left\{ \gamma^{-2}[\gamma t + 1 - \exp(-\gamma t)] - \frac{\kappa^2}{2} \right\}.$$

This expression is valid to second order if the field is relatively weak:

$$N\kappa^2\langle B^\dagger B \rangle \ll 1. \quad (15)$$

In the case of interaction with a single resonant cavity mode, we have $B = ga$ and $\kappa^2\langle B^\dagger B \rangle = n/n_s$, where $n_s = (\gamma_\perp/g)^2$ is a saturation parameter and $n = \langle a^\dagger a \rangle$ is the mean photon number. Then, (15) reduces to the standard condition imposed in the case of resonant coupling between the field and two-level atoms: $Nn/n_s \ll 1$. Solution (14) describes the joint evolution of the atomic ensemble and field starting from an ensemble of ground-state atoms and an arbitrary state of the field.

V. MIXED NONSEPARABLE ATOMIC STATES

Second-order perturbation theory predicts correlation between atoms depending on photon statistics, i.e., provides a framework for describing entangled (inseparable) atomic states. To analyze the properties of the atomic system, we use second-order perturbation theory to find the density matrix for a group of $M \leq N$ atoms, $\rho_A(M \leq N)$, obtained by taking the trace of (14) over the field states represented by ρ_f and over $N - M$ particles. The result has the form

$$\begin{aligned} \rho_A(M \leq N) = & |0\rangle\langle 0|[1 - M\kappa^2\langle B^\dagger B \rangle] + \kappa[\langle B \rangle|1; M\rangle\langle 0| + h.c.] + \kappa^2[\langle B^2 \rangle|2; M\rangle\langle 0| + h.c.] \\ & + \kappa^2\langle B^\dagger B \rangle|1; M\rangle\langle 1; M|. \end{aligned} \quad (16)$$

Note that the density matrix $\rho_A(M \leq N)$ describes a mixed state of the atomic ensemble. Unlike the density matrices for symmetric Dicke states (6), $\rho_A(M \leq N)$ is independent of both N and $p = M/N$. Therefore, the correlations between $M < N$ atoms are identical and are independent of the total particle number. This implies that the state is robust to particle loss.

The atomic density matrix cannot be factorized because of the correlation depending on photon statistics. Consider two atoms described in terms of their respective observables c_1 and c_2 such that $[c_1, c_2] = 0$. Setting $M = 2$ in (16), we have the two-atom density matrix

$$\begin{aligned} \rho_A(2) = & |00\rangle\langle 00|(1 - 2\kappa^2\langle B^\dagger B \rangle) + \kappa\langle B \rangle(|10\rangle\langle 00| + |01\rangle\langle 00| + h.c.) + \kappa^2\langle B^2 \rangle(|11\rangle\langle 00| + h.c.) \\ & + \kappa^2\langle B^\dagger B \rangle(|10\rangle + |01\rangle)(\langle 10| + \langle 01|). \end{aligned} \quad (17)$$

Using (17) we find that the covariance of the operators c_1, c_2 is determined by the electromagnetic field variance:

$$\langle c_1 c_2 \rangle - \langle c_1 \rangle \langle c_2 \rangle = \kappa^2[(\langle B^2 \rangle - \langle B \rangle^2)\langle 0|c_1|1\rangle\langle 0|c_2|1\rangle + (\langle B^\dagger B \rangle - \langle B \rangle \langle B^\dagger \rangle)\langle 1|c_1|0\rangle\langle 0|c_2|1\rangle + c.c.]. \quad (18)$$

If the field is not fluctuating in the sense that its variances are zero, i.e., $\langle B^2 \rangle - \langle B \rangle^2 = 0$ etc. (which is true in the present case, e.g., for a coherent state), then there is no correlation between atoms. Suppose that $c_k (k = 1, 2)$ are dipole operators: $c_k = d_k = \mu(s_{01}(k) + s_{10}(k))$, where the matrix element μ is real. Then the correlation between two dipole moments depends on photon statistics. We define the quadrature operator $X_f = B^\dagger \exp(i\theta) + h.c.$. Then (18) implies that the covariance of the dipole moments is determined by the variance of the quadrature operator normally ordered with respect to the field operators B and B^\dagger at $\theta = 0$: $\langle d_1 d_2 \rangle - \langle d_1 \rangle \langle d_2 \rangle = \mu^2 \kappa^2 D_N$, where $D_N = \langle X_f^2 \rangle - \langle X_f \rangle^2 - \langle [B, B^\dagger] \rangle$. For coherent states, the variance is $D_N = 0$. The dipole moments are correlated both for a squeezed-state field (with $D_N < 0$) and for field in a classical state (with $D_N > 0$).

The necessary and sufficient condition for inseparability of a mixed state is provided by the Peres-Horodecki criterion [9], which is valid for systems with Hilbert spaces of dimension 2×2 and 2×3 . In the case considered here, the state of a two-atom system described by $\rho_A(2)$ is inseparable (entangled) if at least one eigenvalue of the density matrix partially transposed over the variable of atom 1 $\rho_A(2)^{T_1}$ is negative. As example, we consider light in Gaussian and squeezed states.

For a Gaussian field ($\langle B \rangle = \langle B^2 \rangle = 0$), expression (17) reduces to the density matrix describing a superposition of the ground and mixed states: $\rho_A(2) = a|00\rangle\langle 00| + b[(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)]$, where $a + 2b = 1$ and $a = 1 - 2\kappa^2\langle B^\dagger B \rangle$. The eigenvalues of $\rho_A(2)^{T_1}$ are

$$\lambda = \left\{ b, b, \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2} \right\}.$$

Since $\sqrt{a^2/4 + b^2} \approx a/2$, in the approximation considered here, we have the eigenvalues: $\{b, b, a, 0\}$ i.e., a separable state.

Consider the case of resonant interaction with single-mode squeezed light ($B = ga$) generated, for example, by a parametric oscillator. A simple model of the oscillator is defined by the effective Hamiltonian $H = i\hbar(f/2)(a^{\dagger 2} - h.c.)$. The solution is $a = a_0 \cosh r + a_0^\dagger \sinh r$, where $r = f\tau$ is the squeezing parameter, τ is the normalized length of the nonlinear medium, and a_0, a_0^\dagger denote the input field operators. For the initial vacuum state, $\langle a \rangle = 0$, $\langle a^2 \rangle = \langle a^{\dagger 2} \rangle = \cosh r \sinh r$, $\langle a^\dagger a \rangle = \sinh^2 r$. In this case (17) reduces to the following two-atom density matrix

$$\begin{aligned} \rho_A(2) = & |00\rangle\langle 00| [1 - 2\kappa^2 \langle B^\dagger B \rangle] + \kappa^2 \left[\langle B^2 \rangle (|11\rangle\langle 00| + |00\rangle\langle 11|) + h.c. \right] \\ & + \kappa^2 \langle B^\dagger B \rangle (|10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 01|). \end{aligned} \quad (19)$$

The four eigenvalues of $\rho_A^{T_1}(2)$ are

$$\lambda = \left\{ 0; \quad 1 - \frac{2}{n_s} \sinh^2 r; \quad \pm \frac{1}{n_s} \exp(\pm r) \sinh r \right\}. \quad (20)$$

To be specific, we set $r > 0$, i.e., consider the state squeezed with respect to canonical momentum or phase. In this case, $(-1/n_s) \sinh r \exp(-2r) < 0$. However, it is clear that the degree of squeezing is low, because the approximations used here imply that

$$\frac{\sinh^2 r}{n_s} \ll 1. \quad (21)$$

Thus, the state of the atomic system is inseparable. This behavior is explained as follows. Fluctuations of light give rise to correlation between atoms, which leads to two-atom coherence. When condition (21) holds, this coherence plays the key role. Since absorption is weak, the system is almost entirely in the ground state. As distinct to the case of Gaussian statistics, the density matrix has the form $\rho_A(2) \approx |00\rangle\langle 00| + \kappa^2 [\langle B^2 \rangle |11\rangle\langle 00| + h.c.]$.

Note that the following two observations can be inferred from this example. First, a steady entangled atomic state can be created by using weak squeezed light, which looks promising from an experimental perspective. Second, the entire ensemble cannot be interpreted as separable, because any pair in a group of $M \leq N$ atoms is entangled, i.e., the quantum correlation of the ensemble as a whole is robust to particle loss.

Since no reliable universally applicable criterion is known for multiparticle entanglement, we apply the Peres Horodecki criterion to two two-level subsystems and found that any pair of atoms in the ensemble can be inseparable, which gives reason to interpret the state of the entire system as inseparable.

Note also that spurious entanglement may be predicted by perturbation theory [28]. In that study, an example of expansion of the product of two wave functions in terms of a common classical parameter was considered in which individual summands represent entangled states. However, if entanglement entropy is used as a measure, then we have initially independent systems, because the entropy is either quadratic in the small parameter or zero in arbitrary-order perturbation theory. Note that physical implementation of such entangled states, i.e., preparation of an independent state of a pair of entangled particles, requires projective measurement in an entangled basis. The present analysis also relies on perturbation theory, but we deal with a different situation in both physical and formal sense, in which interaction between particles gives rise to correlation. The wavefunction obtained in first-order perturbation theory is not factorizable, and the corresponding entanglement entropy is zero to the corresponding accuracy. This result is physically plausible, because there is no correlation in the first-order perturbation theory. In our analysis, entanglement is predicted by second-order perturbation theory, which describes real emission and absorption processes result in correlation. In this order of perturbation theory, the existence of quantum correlation is substantiated by entanglement criteria consistent with approximation accuracy.

VI. EXACT SOLUTIONS

Radiative decay can be neglected in (2) when evolution over a time $t \ll \gamma^{-1}$ is considered, and the behavior of the entire system is described by the wavefunction $\phi(t) = \exp(-i\hbar^{-1}Ht)(\phi_A \otimes \phi_f)$, where the initial states of the atoms and field are assumed to be uncorrelated. Then, simple solutions can be obtained under certain initial conditions.

Consider the mixing of modes a and b described by

$$H = i\hbar f(a^\dagger b S - ab^\dagger S^\dagger), \quad (22)$$

where $S = S_{10}, S^\dagger = S_{01}$. If analysis is restricted to single-photon Fock states of the modes $\phi_f = c|01\rangle_{ab} + e|10\rangle_{ab}$ exact solutions can be written as

$$\begin{aligned} \exp\{-i\hbar^{-1}Ht\}(c|01\rangle_{ab} + e|10\rangle_{ab}) \otimes \phi_A &= c\left\{|01\rangle \cos[tf\sqrt{SS^\dagger}] + |10\rangle S^\dagger \frac{1}{\sqrt{SS^\dagger}} \sin[tf\sqrt{SS^\dagger}]\right\} \otimes \phi_A \\ &+ e\left\{-|01\rangle S \frac{1}{\sqrt{S^\dagger S}} \sin[tf\sqrt{S^\dagger S}] + |10\rangle \cos[tf\sqrt{S^\dagger S}]\right\} \otimes \phi_A. \end{aligned} \quad (23)$$

In the case of a single-photon process described by the Hamiltonian

$$H = i\hbar g(aS - a^\dagger S^\dagger) \quad (24)$$

there also exist simple solutions. For example,

$$\begin{aligned} &\exp\{-i\hbar^{-1}Ht\}(c|1\rangle \otimes |0; N\rangle + e|0\rangle \otimes |1; N\rangle) \\ &= c\left\{\cos[gf\sqrt{N}]|1\rangle \otimes |0; N\rangle + \frac{1}{\sqrt{N}} \sin[gf\sqrt{N}]|0\rangle \otimes |1; N\rangle\right\} \\ &+ e\left\{-\sqrt{N} \sin[gf\sqrt{N}]|1\rangle \otimes |0; N\rangle + \cos[gf\sqrt{N}]|0\rangle \otimes |1; N\rangle\right\}, \end{aligned} \quad (25)$$

where $|h; N\rangle = |0\rangle^{\otimes N}$, $h = 0, 1$ represents the ground state of the atomic ensemble and a symmetric Dicke state defined in accordance with (6). These solutions are valid only under the restrictions imposed above on the initial states. They describe exchange of excitation between the cavity mode and the atoms.

VII. GENERATION AND TRANSFORMATION OF SYMMETRIC STATES

Now, we use the exact solutions written out above to analyze the evolution of symmetric Dicke states $|h; N\rangle$ in single-photon and wave-mixing processes.

First, consider the case when the spatial inhomogeneity of the field within the region occupied by the atomic ensemble can be neglected. Setting (25) $\phi_A = |h; N\rangle$, we use (9) to obtain

$$\begin{aligned} (\alpha|01\rangle + \beta|10\rangle) \otimes |h; N\rangle &\rightarrow \alpha\left\{\cos\theta_h|01\rangle \otimes |h; N\rangle + \sqrt{\frac{h+1}{N-h}} \sin\theta_h|10\rangle \otimes |h+1; N\rangle\right\} \\ &+ \beta\left\{-\sqrt{\frac{N-h+1}{h}} \sin\theta'_h|01\rangle \otimes |h-1; N\rangle + \cos\theta'_h|10\rangle \otimes |h; N\rangle\right\} \end{aligned} \quad (26)$$

where $\theta_h = tf\sqrt{(h+1)(N-h)}$, $\theta'_h = tf\sqrt{h(N-h+1)}$. Relation (26) entails possibilities of preparation of an entangled from ground-state atoms $|0; N\rangle \rightarrow |1; N\rangle$, and transformation of entangled states by changing the number of excited atoms $|h; N\rangle \rightarrow |h \pm 1; N\rangle$, including disentanglement: $|h; N\rangle \rightarrow |h-1; N\rangle \rightarrow \dots |0; N\rangle$.

Note that exact solutions (25) and (26) describe state swapping, which can be used to map the state of light onto atoms in order to store it in a long-lived atomic ensemble, i.e., to implement quantum memory. In particular, an unknown superposition of photons can be transferred to atoms and back by using the following transformation entailed by (25)

$$(\alpha|1\rangle + \beta|0\rangle) \otimes |0; N\rangle \rightleftharpoons |0\rangle \otimes \left(\alpha \frac{1}{\sqrt{N}}|1; N\rangle + \beta|0; N\rangle\right). \quad (27)$$

Solutions (25) and (26) make it possible to take into account the spatial configuration of atoms in the ensemble. For example, consider the interaction between a one-dimensional array of atoms located at points x_1, \dots, x_N and a single photon described by Hamiltonian (24) with $S = \sum_p s_{10}(p) \exp[ikx_p]$, where $s_{10}(p) = |1\rangle_p \langle 0|$ corresponds to the atom located at x_p , $p = 1, \dots, N$. Using (25), we can show that

$$|1\rangle \otimes |0; N\rangle \rightarrow \cos\theta|1\rangle \otimes |0; N\rangle + \sin\theta|0\rangle \otimes \eta_N(1), \quad (28)$$

$$\theta = tg\sqrt{N},$$

$$\eta_N = (1/\sqrt{N}) \left[e^{ikx_1} |10\dots 0\rangle + \dots e^{ikx_N} |0\dots 01\rangle \right]. \quad (29)$$

Expression (29) implies that an array of entangled atoms is created when $\theta = \pi/2$. Note that η_N is the Dicke state with $j = m = N/2 - 1$ only if $\sum_p \exp[ikx_p] = 0$.

Solutions (23), (25) imply that a photon and an atomic ensemble are entangled via interaction. If photons are entangled (e.g., by projective measurement) in a combination of such independent systems, then the atomic ensembles will become entangled. We consider optical measurement schemes based on this method, known as entanglement swapping. The key resources used in these schemes are set of atomic ensembles correlated with respective photons, beamsplitters, and single-photon detectors. The analysis that follows is restricted to schemes in which only specific single-photon output is recorded.

As an initial state, we use the EPR pair

$$Z(W) = a|0\rangle_f \otimes |0\rangle + b|1\rangle_f \otimes |W\rangle, \quad (30)$$

where Fock states are denoted by the subscript f, $|W\rangle = |1; N\rangle/\sqrt{N}$, $|0\rangle = |0; N\rangle$. It is generated by the mode mixing described by (22), where the mode b is a classical wave. The state of n independent identical ensembles entangled with respective photons is represented by the product

$$Z_n(W) = Z(W)^{\otimes n} = a^{n-1}b \left[|10\dots 0\rangle_f \otimes |W0\dots 0\rangle + \dots |00\dots 1\rangle_f \otimes |00\dots W\rangle \right] + \dots \quad (31)$$

As illustrated by the figure, the photons associated with atomic ensembles are injected into a system of $n - 1$ beamsplitters with n input ports and n output ports. Each beamsplitter performs the transformation $|01\rangle_f \rightarrow$

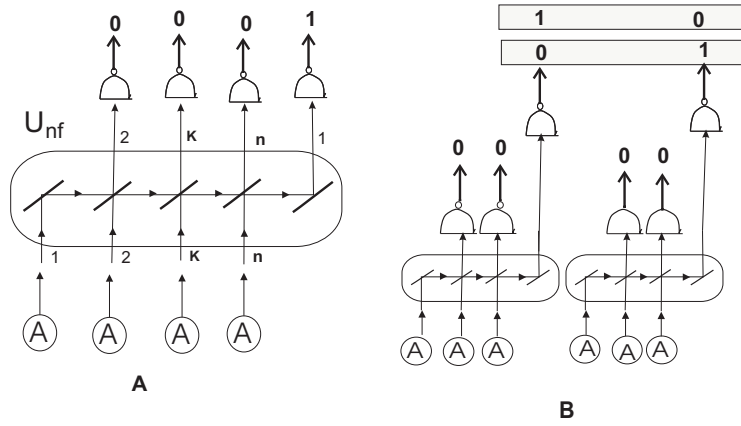


FIG. 1: (a) Scheme for generating entangled states of atomic ensembles. (b) Preparation of entangled states by correlation of photocounts recorded by two schemes.

$c_k|01\rangle_f + s_k|10\rangle_f$, $|10\rangle_f \rightarrow -s_k|01\rangle_f + c_k|10\rangle_f$, where $c_k^2 + s_k^2 = 1$, $k = 1 \dots n - 1$. The scheme is described by a unitary operator U_{nf} and characterized by the following property. There exist an input port optically coupled to every output port and an output port optically coupled to every input port. In Fig. 1a, the latter is output port 1. We call it the optical output port, and the corresponding detector is called the output detector. The scheme performs the transformation

$$\begin{aligned} U_{nf}|1\dots 0\rangle_f &= t_1|1\dots 0\rangle_f + \dots + t_n|0\dots 1\rangle_f, \\ U_{nf}^{-1}|1\dots 0\rangle_f &= \tau_1|1\dots 0\rangle_f + \dots + \tau_n|0\dots 1\rangle_f, \end{aligned} \quad (32)$$

where the coefficients t_k, τ_k , $k = 1, \dots, n$ are determined by the transmittances and reflectances of the beamsplitters, and $\sum_k t_k^2 = \sum_k \tau_k^2 = 1$.

If output detector detects a photon which corresponds to the state $|1_f\rangle = |1\dots 0\rangle$, then with the probability

$$Prob(1) = |a^{n-1}b|^2 \quad (33)$$

entangled state of atomic ensembles will be prepared

$$\begin{aligned} \langle 1_f | U_{nf} Z_n(W) / \sqrt{Prob(1)} &= \eta_n(W), \\ \eta_n(W) &= q_1|W\dots 0\rangle + q_n|0\dots W\rangle, \end{aligned} \quad (34)$$

This scheme has the following property. Since the coefficients q_1, \dots, q_n are completely determined by the transmittances and reflectances of the beamsplitters, weakly entangled states $Z(W)$ can be used to prepare highly entangled states atomic ensembles.

Let us consider several particular cases. If $n = 2$, then $q_1 = c_1, q_2 = s_1$, and we have an EPR pair of the form $\eta_2(W) = EPR(W) = c_1|W0\rangle + s_1|0W\rangle$. When $c_1 = s_1 = 1/\sqrt{2}$ it is maximally entangled. If $n = 3$ and $q_1 = c_1c_2, q_2 = -s_1c_2, q_3 = s_2$, then we have a W state. If $c_1 = -s_1 = 1/\sqrt{2}, c_2 = \sqrt{2/3}$ and $c_3 = \sqrt{2/3}$, then

$$\eta_3(W) = W(W) = (1/\sqrt{3})(|W00\rangle + |0W0\rangle + |00W\rangle). \quad (35)$$

In particular, one can prepare the asymmetric state $\widetilde{W}(W) = (1/\sqrt{2})|W00\rangle + (1/2)|0W0\rangle + (1/2)|00W\rangle$. When $N = 1$, it is unitary equivalent to the GHZ state and can be used as a quantum channel for teleportation or dense coding [29].

Using correlation between photocounts in a combination of schemes considered above, mixed states of atomic ensembles can be prepared, including inseparable ones. For example, consider two independent identical schemes $S_2(X)$ combined as shown in Fig. 1b, with three single-photon detectors in each scheme. If a photon is detected by either scheme, then we have the pair of states $\langle 1_f | S_2(X) \otimes \langle 0_f | S_2(X) w = |\eta_2(X), 0\rangle$ and $\langle 0_f | S_2(X) \otimes \langle 1_f | S_2(X) w = |0, \eta_2(X)\rangle$. Suppose that the detector outputs are connected so that a single photon produced by either scheme is counted. This measurement is described by the projector $|1_f 0_f\rangle\langle 1_f 0_f| + |0_f 1_f\rangle\langle 0_f 1_f|$. The resulting mixed state is represented by a density matrix of the form

$$\rho(X) = (1/2) \left[|\eta_2(X), 0\rangle\langle \eta_2(X), 0| + |0, \eta_2(X)\rangle\langle 0, \eta_2(X)| \right]. \quad (36)$$

Its separability is an open question, because a necessary and sufficient condition is known only for mixed systems of dimension $2 \times 2, 2 \times 3$. However, if we assume that $N = 1$, i.e., consider a combination of four atoms instead of ensembles, then $\eta_2(X) = \Psi^+ = (1/\sqrt{2})(|01\rangle + |10\rangle)$ and density matrix (36) describes a four-particle state:

$$\rho(4) = (1/2)(|\Psi^+00\rangle\langle \Psi^+00| + |00\Psi^+\rangle\langle 00\Psi^+|). \quad (37)$$

Taking the state of the pair of atoms in the first scheme defined by the two-particle reduced density matrix $\rho(2) = (1/2) \left[|\Psi^+\rangle\langle \Psi^+| + |00\rangle\langle 00| \right]$, we can apply the separability criterion. The density matrix partially transposed over the variables of the one atom has four eigenvalues one of which is negative $1/4, 1/4, (1 \pm \sqrt{2})/4$. Therefore, the density matrix $\rho(4)$ is inseparable.

IX. HIERARCHIC STRUCTURE OF STATES

Note that expression (34) is hierarchically structured. To illustrate this property, we consider a combination of schemes generating states of this type. As distinct to schemes using correlation of photocounts, we consider optically connected schemes. If an elementary scheme that performs the transformation $S_n(X) = U_{nf}Z_n(W)$ with $X = W$ (see Fig. 1a) records single-photon output, then the resulting state has the form of (34):

$$\langle 1_f | S_n(X) w = \eta_n(X) = \tau_1^i |X0 \dots 0\rangle + \dots + \tau_n^i |00 \dots X\rangle, \quad (38)$$

where $w = 1/\sqrt{\text{Prob}(1)}$. We define the optical output port of the scheme $S_n(X)$ as the one optically coupled to every input port. In Fig. 1a, it is output port 1. The input port of the scheme $S_n(X)$ is defined as the optical input port of the system of beamsplitters. Then, we can take, for example, p independent schemes represented as $(S_n(X))^p$ and use their optical outputs as the input of the scheme S_p . As a result, we have a new scheme $S_p((S_n(X))^p)$. If it records single-photon output, we have an entangled state that consists of lower level entangled states:

$$\langle 1_f | S_p((S_n(X))^p) w = \eta_p(\eta_n(X)) = t_1 |\eta_n(X), 0 \dots 0\rangle + \dots + t_p |0, 0, \dots \eta_n(X)\rangle. \quad (39)$$

By virtue (38) it takes the forme:

$$\eta_p(\eta_n(X)) = \eta_{pn}(X). \quad (40)$$

Thus, we can formulate the following property. The state $\eta_n(X)$ defined by (35) with $n = n_1 n_2 \dots n_p$ can be represented as

$$\eta_n(X) = \eta_{n_1}(\eta_{n_2}(\dots(\eta_{n_p}))). \quad (41)$$

This implies that the vector $\eta_n(X)$ has the structure of an entangled state with respect to any group of s particles, where s is such that n/s is a natural number greater than unity.

When the wavefunction $\eta_n(X)$ is symmetric, a hierarchically structured representation can be obtained by using the permanent expansion defined as a determinant with a summation rule for permutations depending on symmetry [30]. In particular, successive decomposition of a determinant with respect to rows or columns and subsequent association of summands can be used to represent a permanent in terms of permanents of lower dimension, which reflects hierarchical structure.

For example, when $n = 6$, it holds that

$$\eta_6(X) = \eta_3(\eta_2(X)) = \eta_2(\eta_3(X)). \quad (42)$$

This state has the structure of an EPR pair or a W state:

$$\eta_3(\eta_2(X)) = W(EPR) = EPR(W).$$

This example demonstrates that the same state exhibits structure characteristic of entangled states of two different types. This property can be used in different applications: the EPR pair can serve as a quantum channel for teleportation or dense coding, while the symmetric W state can be used for cloning.

To choose a particular structure defined by the dimension of the Hilbert space of its element, appropriate basis vectors and observables should be introduced. In physical terms, this is equivalent to a two-level approximation. Indeed, any group of s particles, where s is such that n/s is a natural number greater than unity, is represented in $\eta_n(X)$ by two states, $|0\rangle = 0_s$, $\eta_s(X) = 1_s$. The group can be treated as a two-level particle (qubit) with basis vectors 0_s and 1_s . Such qubits and hierarchically structured states $\eta_n(X)$ can be used in quantum information processing. By analogy with (29), the vector $\eta_n(X)$ represents a Dicke state only if $\eta_n(X)$.

X. CONCLUSIONS

A model describing resonant interaction of identical two-level atoms with a narrow-band radiation field is used to analyze multiparticle entanglement. The interaction is described by an effective Hamiltonian that allows for various multiphoton processes. The statistics of radiation and atoms are characterized by a density matrix, for which solutions are calculated in secondorder perturbation theory in the interaction strength and exact solutions are found in the case of negligible decay. It is shown that the state of any pair of atoms interacting with weak single-mode squeezed light is inseparable and robust against decay. It is demonstrated that symmetric entangled multiparticle states can be generated by using optical schemes based on singlephoton projection. An optical scheme is described that can be used to prepare highly states of entangled atomic ensembles from weakly entangled states by projective measurement.

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